

# A NOTE ON METRIC INHOMOGENEOUS DIOPHANTINE APPROXIMATION

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ABSTRACT. Suppose that  $\varphi(n)$  is a monotone increasing function such that  $\sum_n 1/(n\varphi(n))$  diverges. We consider conditions of an irrational  $\theta$  for which

$$\liminf_{n \rightarrow \infty} n\varphi(n)\|n\theta - s\| = 0 \text{ for almost every } s.$$

For a certain class of irrationals the divergence of  $\sum_n 1/(n\varphi(n))$  guarantees the limit inferior is 0. But this is not true for general irrational  $\theta$ . If  $\varphi(n)$  increases to infinity, there exist an irrational  $\theta$  for which the limit inferior is infinity.

## 1. INTRODUCTION

The inhomogeneous Diophantine approximation theorem by Minkowski[8] state that for an irrational number  $\theta$ , if  $s$  is not of the form  $B\theta - A$  for integers  $A$  and  $B$ , then there are infinitely many integer  $n$  such that

$$\|n\theta - s\| < \frac{1}{4|n|}$$

where  $\|t\|$ ,  $t \in \mathbb{R}$  be the distance to its nearest integer.

Let  $I_n$  be a sequence of intervals such that  $|I_n| \rightarrow 0$  and  $J_n$  be a sequence of measurable set such that  $J_n \subset I_n$  and  $\mu(J_n) > c\mu(I_n)$ , where  $\mu$  denote the Lebesgue measure. Cassels[1] showed that (See [3], p.29)

$$(1.1) \quad \mu\left(\bigcap_{k \geq 1} \bigcup_{n \geq k} I_n\right) = \mu\left(\bigcap_{k \geq 1} \bigcup_{n \geq k} J_n\right).$$

Therefore, for any irrational  $\theta$  and a constant  $c$  we have for almost every  $s \in \mathbb{R}$

$$(1.2) \quad \|n\theta - s\| < \frac{c}{n} \quad \text{for infinitely many } n \in \mathbb{N}$$

(See also [6]).

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The first Borel-Cantelli lemma implies that if  $\sum_n \psi(n) < \infty$ , then for almost every  $s \in \mathbb{R}$  we have

$$\|n\theta - s\| < \psi(n) \quad \text{for finitely many } n \in \mathbb{N}.$$

When  $\sum_n \psi(n) = \infty$  and  $\psi(n) = o(1/n)$ , we have for almost every  $s \in \mathbb{R}$

$$\|n\theta - s\| < \psi(n) \quad \text{for infinitely many } n \in \mathbb{N}$$

according to the irrational  $\theta$  and  $\psi(n)$ . In this paper, we investigate the conditions for the irrational  $\theta$  and  $\psi(n)$ . More specifically, we put  $\psi(n) = \frac{1}{n\varphi(n)}$  and consider the condition for

$$(1.3) \quad \liminf_{n \rightarrow \infty} n\varphi(n) \cdot \|n\theta - s\| = 0 \quad \text{almost every } s \in \mathbb{R}.$$

We only consider  $\varphi(n)$  be a monotone increasing function with

$$\sum_{n=1}^{\infty} \frac{1}{n\varphi(n)} = \infty.$$

Note that the limit inferior of (1.3) is 0 or infinity for almost every  $s$  by (1.1).

An irrational  $\theta$  is said to be bounded type if there exist a  $C > 0$  such that  $n\|n\theta\| > C$  for all positive integer  $n$ . Kurzweil[7] showed that, if and only if the irrational  $\theta$  is of bounded type, then for almost every  $s$  and a monotone decreasing positive function  $\psi$  with  $\sum \psi(n) = \infty$ ,

$$(1.4) \quad \|n\theta - s\| < \psi(n) \quad \text{for infinitely many } n \in \mathbb{N}$$

holds. (See also [2] for the higher dimensional case). He also showed that for a given monotone decreasing  $\psi(n)$  the set of irrationals  $\theta$  for (1.4) holds for almost every  $s$  has full measure.

Since we assume the monotonicity of  $\varphi(n)$  which is a little stronger than the monotonicity of  $\psi(n)$ , for a slightly wider class of irrational  $\theta$ , (1.3) holds for any monotone  $\varphi(n)$  with diverging  $\sum \frac{1}{n\varphi(n)}$  (See Proposition 2.5). For a general irrational  $\theta$ , (1.3) does not hold for any  $\varphi(n)$  and the condition for (1.3) should depend on Diophantine type of the irrational  $\theta$ . (See Theorem 2.2 and Theorem 2.4) Moreover, if  $\varphi(n)$  goes to infinity, then (1.3) cannot be true for every irrational  $\theta$ . (Corollary 2.3)

## 2. MAIN THEOREMS

For an irrational number  $0 < \theta < 1$ , we have a unique continued fraction expansion with partial quotients  $a_k$ ,  $k \geq 1$ . Let  $p_k/q_k$  be the  $k$ -th convergent and put  $p_0 = 0$  and  $q_0 = 1$ .  $q_{i+1} = a_{i+1}q_i + q_{i-1}$ ,  $p_{i+1} = a_{i+1}p_i + p_{i-1}$  and  $\|q_i\theta\| = a_{i+2}\|q_{i+1}\theta\| + \|q_{i+2}\theta\|$  for  $i \geq 1$ . For more details, consult [5] and [9].

We have the main theorems of this paper as follows: The first theorem presents a condition for the infinite lower limit of (1.3).

**Theorem 2.1.** *Let  $\psi(n)$  be a monotone decreasing function and  $\theta$  be an irrational  $\theta$  with principal convergents  $p_k/q_k$ . Then for almost every  $x \in \mathbb{R}$ ,*

$$\|n\theta - s\| < \psi(n) \quad \text{for finitely many } n \in \mathbb{N}$$

if

$$\sum_{k=0}^{\infty} \left( \sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k\theta\|) \right) < \infty.$$

If  $\psi(n)$  is in a special form, then we have a little easier characterization.

**Theorem 2.2.** *Let  $\varphi(n)$  be a monotone increasing function. For a given irrational  $\theta$  we have*

$$\liminf_{n \rightarrow \infty} n\varphi(n) \cdot \|n\theta - s\| = \infty \quad \text{almost every } s$$

if the convergent's denominator  $q_k$  of the irrational  $\theta$  satisfies

$$\sum_{k=0}^{\infty} \frac{\log(\min(\varphi(q_k), q_{k+1}/q_k))}{\varphi(q_k)} < \infty.$$

For any monotone increasing  $\varphi(n)$ , choose an irrational  $\theta$  such that  $\varphi(q_k) > k^2$ . Then

$$\sum_{k=0}^{\infty} \frac{\log(\min(\varphi(q_k), q_{k+1}/q_k))}{\varphi(q_k)} \leq \sum_{k=0}^{\infty} \frac{\log \varphi(q_k)}{\varphi(q_k)} < \sum_{k=0}^{\infty} \frac{2 \log k}{k^2} < \infty.$$

Therefore, we have the following corollary:

**Corollary 2.3.** *For any monotone increasing function  $\varphi(n)$  which goes to infinity, there exist an irrational  $\theta$  such that*

$$\liminf_{n \rightarrow \infty} n\varphi(n) \cdot \|n\theta - s\| = \infty \quad \text{almost every } s.$$

Also we have a theorem for a sufficient condition for (1.3).

**Theorem 2.4.** *Let  $\varphi(n)$  be a monotone increasing function. For the irrational  $\theta$  with the convergent's denominator  $q_k$  such that  $\sum_{k=0}^{\infty} 1/\varphi(q_k) = \infty$  we have*

$$\liminf_{n \rightarrow \infty} n\varphi(n) \cdot \|n\theta - s\| = 0 \quad \text{almost every } s.$$

Note that if  $\varphi(n)$  is bounded, then  $\sum_k 1/\varphi(q_k) = \infty$  and we conclude (1.2). Using the monotonicity condition of  $\varphi(n)$ , we have the Kurzweil type theorem (1.4) for a little bigger set of irrationals than irrationals of bounded type.

**Proposition 2.5.** *Let  $\theta$  be an irrational with  $q_k \leq C^k$  for some constant  $C$  and  $\varphi(n)$  be a monotone increasing function. Then we have*

$$\liminf_{n \rightarrow \infty} n\varphi(n) \cdot \|n\theta - s\| = 0 \quad \text{almost every } s$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n\varphi(n)} = \infty.$$

The following proposition gives a necessary condition of the equivalence between 0 limit inferior and the divergence of  $\sum_n \frac{1}{n\varphi(n)}$ .

**Proposition 2.6.** *Let  $\theta$  be an irrational such that*

$$\sum_{k=1}^{\infty} \frac{1}{\log q_k} < \infty.$$

*Then there is a monotone increasing  $\varphi(n)$  such that  $\sum_{n=1}^{\infty} \frac{1}{n\varphi(n)} = \infty$  and*

$$\liminf_{n \rightarrow \infty} n\varphi(n) \cdot \|n\theta - s\| = \infty \quad \text{almost every } s.$$

The proof of Theorem 2.1 and Theorem 2.2 are given in Section 3. In Section 4, Theorem 2.4 is shown. Proposition 2.5 and Proposition 2.6 are proved in the last section.

**Example 2.7.** Let  $\varphi(n) = \log n$ . Then, clearly,  $\sum_{n=2}^{\infty} \frac{1}{n\varphi(n)} = \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty$ . Choose an irrational  $\theta$  with  $q_k > \exp(k^{1+\epsilon})$ . Then  $\sum_{k=2}^{\infty} \frac{\log(\min(\varphi(q_k), q_{k+1}/q_k))}{\varphi(q_k)} \leq \sum_{k=2}^{\infty} \frac{\log(\log q_k)}{\log q_k} < \sum_{k=2}^{\infty} \frac{(1+\epsilon) \log k}{k^{1+\epsilon}} < \infty$  and by Theorem 2.2 we have

$$\liminf_{n \rightarrow \infty} n \log n \cdot \|n\theta - s\| = \infty \quad \text{almost every } s.$$

However, for an irrational  $\theta$  which satisfies  $q_k < Ck^k$ , we have  $\sum_{k=2}^{\infty} \frac{1}{\log q_k} > \sum_{k=2}^{\infty} \frac{1}{k \log k + \log C} = \infty$  and from Theorem 2.4

$$\liminf_{n \rightarrow \infty} n \log n \cdot \|n\theta - s\| = 0 \quad \text{almost every } s.$$

## 3. SUFFICIENT CONDITION

In this section, we give sufficient conditions of (1.3). These are Theorem 2.1 and Theorem 2.2.

Let  $B(x, r)$  be the ball centered at  $x$  with radius  $r$ . Denote

$$F_k = \bigcup_{0 \leq n < q_k} B(n\theta, \psi(n))$$

and

$$E_k = F_k \setminus F_{k-1}.$$

*Proof of Theorem 2.1.* Since  $\|n\theta - (n - q_k)\theta\| = \|q_k\theta\|$  and  $\psi(n)$  is monotone decreasing, we have for each  $q_k \leq n < q_{k+1}$

$$\mu\left(B(n\theta, \psi(n)) \setminus B(n - q_k)\theta, \psi(n - q_k)\right) \leq \|q_k\theta\|.$$

Clearly we also have that for each  $q_k \leq n < q_{k+1}$

$$\mu\left(B(n\theta, \psi(n)) \setminus B(n - q_k)\theta, \psi(n - q_k)\right) \leq \mu(B(n\theta, \psi(n))) = 2\psi(n).$$

Hence, we have

$$\begin{aligned} \mu(E_{k+1}) &= \mu(F_{k+1} \setminus F_k) \\ &\leq \mu\left(\bigcup_{n=q_k}^{q_{k+1}-1} \left(B(n\theta, \psi(n)) \setminus B((n - q_k)\theta, \psi(n - q_k))\right)\right) \\ &\leq \sum_{n=q_k}^{q_{k+1}-1} \mu\left(B(n\theta, \psi(n)) \setminus B((n - q_k)\theta, \psi(n - q_k))\right) \\ &\leq \sum_{n=q_k}^{q_{k+1}-1} \min(\|q_k\theta\|, 2\psi(n)). \end{aligned}$$

By the first Borel-Cantelli lemma, we complete the proof of Theorem 2.1.  $\square$

*Proof of Theorem 2.2.* Let

$$\psi(n) = \frac{1}{n\varphi(n)}.$$

Then since  $\varphi(n)$  is monotone increasing, we have

$$\begin{aligned} (3.1) \quad \sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k\theta\|) &\leq \sum_{n=q_k}^{q_{k+1}-1} \psi(n) \leq \sum_{n=q_k}^{q_{k+1}-1} \frac{1}{n\varphi(n)} dx = \int_{q_k}^{q_{k+1}} \frac{dx}{x\varphi(x)} \\ &= \int_{\log q_k}^{\log q_{k+1}} \frac{dt}{\varphi(e^t)} \leq \frac{\log(q_{k+1}/q_k)}{\varphi(q_k)}. \end{aligned}$$

If  $\varphi(q_k)q_k < q_{k+1}$ , then we have

$$\begin{aligned}
 \sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k \theta\|) &= \sum_{n=q_k}^{\lceil q_{k+1}/\varphi(q_k) \rceil - 1} \|q_k \theta\| + \sum_{n=\lceil q_{k+1}/\varphi(q_k) \rceil}^{q_{k+1}-1} \psi(n) \\
 &\leq \left( \left\lceil \frac{q_{k+1}}{\varphi(q_k)} \right\rceil - q_k \right) \|q_k \theta\| + \int_{\lceil q_{k+1}/\varphi(q_k) \rceil}^{q_{k+1}} \frac{1}{n\varphi(n)} dx \\
 (3.2) \quad &\leq \frac{q_{k+1}}{\varphi(q_k)} \|q_k \theta\| + \int_{q_{k+1}/\varphi(q_k)}^{q_{k+1}} \frac{dx}{x\varphi(x)} \\
 &= \frac{1}{\varphi(q_k)} + \int_{\log(q_{k+1}/\varphi(q_k))}^{\log(q_{k+1})} \frac{dt}{\varphi(e^t)} \\
 &\leq \frac{1}{q_k} + \frac{\log \varphi(q_k)}{\varphi(q_{k+1}/\varphi(q_k))} \leq \frac{1}{q_k} + \frac{\log \varphi(q_k)}{\varphi(q_k)}.
 \end{aligned}$$

By (3.1) and (3.2), if

$$\sum_{k=1}^{\infty} \frac{\log \min(\varphi(q_k), q_{k+1}/q_k)}{\varphi(q_k)} < \infty,$$

then

$$\sum_{k=0}^{\infty} \left( \sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k \theta\|) \right) < \infty.$$

Therefore, by Theorem 2.1, the proof Theorem 2.2 is obtained.  $\square$

#### 4. NECESSARY CONDITION

In this section, we consider a necessary condition of (1.3).

For  $k \geq 1$  let

$$F_k = \bigcup_{0 \leq n < q_k} B\left(n\theta, \frac{1}{n\varphi(n)}\right)$$

and

$$G_k = \bigcup_{0 \leq n < q_k} B\left(n\theta, \frac{1}{q_k \varphi(q_k)}\right).$$

Then clearly we have  $G_k \subset F_k$ . If  $\varphi(q_k) \geq 4$ , then each balls in  $G_k$  are disjoint and

$$\mu(G_k) = q_k \cdot \frac{2}{q_k \varphi(q_k)} = \frac{2}{\varphi(q_k)}.$$

Now we estimate  $\mu(G_\ell \cap G_k)$ ,  $\ell < k$  by the Denjoy-Koksma inequality (see e.g., [4]): Let  $T$  be an irrational rotation by  $\theta$  and  $f$  be a real valued function of bounded variation on the unit interval. Then for any  $x$  we have

$$(4.1) \quad \left| \sum_{n=0}^{q_k-1} f(T^n x) - q_k \int f d\mu \right| < \text{var}(f).$$

For a given  $m\theta$ , by the Denjoy-Koksma inequality (4.1) we have

$$\begin{aligned} \#\left\{0 \leq n < q_k \mid n\theta \in B\left(m\theta, \frac{1}{q_\ell \varphi(q_\ell)}\right)\right\} &= \sum_{n=0}^{q_k-1} 1_{B(m\theta, 1/(q_\ell \varphi(q_\ell)))(T^n x)} \\ &< q_k \cdot \frac{2}{q_\ell \varphi(q_\ell)} + 2. \end{aligned}$$

Since  $G_k$  consists of the intervals of centered at  $n\theta$ ,  $0 \leq n < q_k$  with radius  $(q_k \varphi(q_k))^{-1}$ , we have for each  $m$  with  $0 \leq m < q_\ell$

$$\mu\left(G_k \cap B\left(m\theta, \frac{1}{q_\ell \varphi(q_\ell)}\right)\right) < \left(\frac{2q_k}{q_\ell \varphi(q_\ell)} + 3\right) \frac{2}{q_k \varphi(q_k)}.$$

Therefore we have for  $k > \ell$

$$\begin{aligned} \mu(G_k \cap G_\ell) &< q_\ell \left(\frac{2q_k}{q_\ell \varphi(q_\ell)} + 3\right) \frac{2}{q_k \varphi(q_k)} = \mu(G_k)\mu(G_\ell) + \frac{3q_\ell}{q_k} \mu(G_k) \\ &< \mu(G_k)\mu(G_\ell) + 3 \left(\frac{1}{2}\right)^{\lfloor (k-\ell)/2 \rfloor} \mu(G_k) \leq \mu(G_k)\mu(G_\ell) + \frac{3\sqrt{2}}{2^{(k-\ell)/2}} \mu(G_k). \end{aligned}$$

We need a version of Borel-Cantelli lemma (e.g. [10]) to go further:

**Lemma 4.1.** *Let  $(\Omega, \mu)$  be a measure space, let  $f_k(\omega)$  ( $k = 1, 2, \dots$ ) be a sequence of nonnegative  $\mu$ -measurable functions, and let  $f_k, \varphi_k$  be sequences of real numbers such that*

$$0 \leq f_k \leq \varphi_k \leq 1 \quad (k = 1, 2, \dots).$$

*Suppose that*

$$\int_{\Omega} \left( \sum_{m < k \leq n} f_k(\omega) - \sum_{m < k \leq n} f_k \right)^2 d\mu \leq C \sum_{m < k \leq n} \varphi_k$$

*for arbitrary integers  $m, n$  ( $m < n$ ). Then*

$$\sum_{1 \leq k \leq n} f_k(\omega) = \sum_{1 \leq k \leq n} f_k + O(\Phi^{1/2}(n) \ln^{3/2+\varepsilon} \Phi(n))$$

*for almost all  $\omega \in \Omega$ , where  $\varepsilon > 0$  is arbitrary and  $\Phi(n) = \sum_{1 \leq k \leq n} \varphi_k$ .*

Put  $f_k = \varphi_k = \mu(G_k)$  and  $f_k(x) = 1_{G_k}(x)$  in Lemma 4.1. Then we have for any  $m < n$

$$\begin{aligned} &\int \left( \sum_{m < k \leq n} f_k(\omega) - \sum_{m < k \leq n} f_k \right)^2 d\mu \\ &\leq 2 \sum_{m < \ell < k \leq n} (\mu(G_k \cap G_\ell) - \mu(G_k)\mu(G_\ell)) + \sum_{m < k \leq n} \mu(G_k) \\ &< 2 \sum_{m < k \leq n} \sum_{m < \ell < k} \frac{3\sqrt{2}}{2^{(k-\ell)/2}} \mu(G_k) + \sum_{m < k \leq n} \mu(G_k) < \left( \frac{12}{\sqrt{2}-1} + 1 \right) \sum_{m < k \leq n} \mu(G_k). \end{aligned}$$

Therefore, by Lemma 4.1, if

$$\sum_k \mu(G_k) = \sum_k \frac{2}{\varphi(q_k)} = \infty,$$

then we have for almost every  $x$

$$\sum_{k=1}^{\infty} 1_{G_k}(x) = \infty$$

or

$$x \in G_k \subset F_k \text{ infinitely many } k' \text{'s.}$$

Hence, we have the proof of Theorem 2.4.

## 5. CONDITIONS FOR THE GENERALIZED KURZWEIL TYPE THEOREM.

In this section we give the proof of Proposition 2.5 and 2.6.

*Proof of Proposition 2.5.* Suppose that  $q_k \leq C^k$  and  $\varphi(n)$  be monotone increasing with  $\sum_n \frac{1}{n\varphi(n)} = \infty$ . Let  $\varphi(x) = \varphi(\lfloor x \rfloor)$  be a function defined on real  $x \geq 1$ . Then we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{\varphi(q_k)} &\geq \sum_{k=0}^{\infty} \frac{1}{\varphi(C^k)} = \sum_{k=0}^{\infty} \frac{1}{\log C} \int_{k \log C}^{(k+1) \log C} \frac{dt}{\varphi(C^k)} \\ &\geq \frac{1}{\log C} \sum_{k=0}^{\infty} \int_{k \log C}^{(k+1) \log C} \frac{dt}{\varphi(e^t)} = \frac{1}{\log C} \int_0^{\infty} \frac{dt}{\varphi(e^t)} \\ &= \frac{1}{\log C} \int_1^{\infty} \frac{dx}{x\varphi(x)} = \frac{1}{\log C} \sum_{n=1}^{\infty} \frac{1}{n\varphi(n)} = \infty. \end{aligned}$$

By Theorem 2.4 we complete the proof.  $\square$

*Proof of Proposition 2.6.* Let  $\varphi(n) = \log n \cdot \log(\log n)$  for large  $n$ . Then for large  $M$  we have

$$\begin{aligned} \sum_{k=M}^{\infty} \frac{\log(\min(\varphi(q_k), q_{k+1}/q_k))}{\varphi(q_k)} &\leq \sum_{k=M}^{\infty} \frac{\log \varphi(q_k)}{\varphi(q_k)} \\ &= \sum_{k=M}^{\infty} \frac{\log(\log q_k) + \log(\log(\log q_k))}{\log q_k \cdot \log(\log q_k)} \\ &= \sum_{k=M}^{\infty} \frac{1 + \log(\log(\log q_k)) / \log(\log q_k)}{\log q_k} \\ &\leq \sum_{k=M}^{\infty} \frac{2}{\log q_k} < \infty. \end{aligned}$$

By Theorem 2.2 we complete the proof.  $\square$



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## REFERENCES

- [1] J.W.S. Cassels, *Some metrical theorems in Diophantine approximation I*, Proc. Cambridge Philos. Soc. **46**, (1950), 209-218.
- [2] B. Fayad, *Mixing in the absence of the shrinking target property*, Bull. London Math. Soc. **38** (2006), no. 5, 829-838.
- [3] G. Harman, *Metric Number Theory*, Oxford Univ. Press, 1998.
- [4] M.R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Inst. Hautes Études Sci. Publ. Math. **49** (1979), 5-233.
- [5] A. Ya. Khinchin, *Continued fractions*, Univ. Chicago Press, Chicago, 1964.
- [6] D.H. Kim, *The shrinking target property of irrational rotations*, Nonlinearity **20** (2007), 1637-1643.
- [7] J. Kurzweil, *On the metric theory of inhomogeneous diophantine approximations*, Studia Math. **15** (1955), 84-112.
- [8] H. Minkowski, *Diophantische Approximationen*, Teubner, Leipzig, 1907.
- [9] A. Rockett and P. Szűsz, *Continued Fractions*, World Scientific, Singapore, 1992.
- [10] V. Sprindžuk, *Metric Theory of Diophantine Approximations*, V. H. Winston & Sons, Washington, D.C., 1979.

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